Exercise 42

Calculate y'.

$$y = \frac{(x+\lambda)^4}{x^4+\lambda^4}$$

Solution

Take the logarithm of both sides to simplify the right side.

$$\begin{aligned}\ln y &= \ln \frac{(x+\lambda)^4}{x^4+\lambda^4} \\ &= \ln(x+\lambda)^4 - \ln(x^4+\lambda^4) \\ &= 4\ln(x+\lambda) - \ln(x^4+\lambda^4)\end{aligned}$$

Take the derivative of both sides with respect to x.

$$\begin{aligned}\frac{d}{dx}(\ln y) &= \frac{d}{dx}\left[4\ln(x+\lambda) - \ln(x^4+\lambda^4)\right] \\ \frac{1}{y}\cdot\frac{dy}{dx} &= 4\left[\frac{d}{dx}\ln(x+\lambda)\right] - \left[\frac{d}{dx}\ln(x^4+\lambda^4)\right] \\ \frac{1}{y}\frac{dy}{dx} &= 4\left(\frac{1}{x+\lambda}\right)\cdot\frac{d}{dx}(x+\lambda) - \left(\frac{1}{x^4+\lambda^4}\right)\cdot\frac{d}{dx}(x^4+\lambda^4) \\ &= 4\left(\frac{1}{x+\lambda}\right)\cdot(1) - \left(\frac{1}{x^4+\lambda^4}\right)\cdot(4x^3) \\ &= \frac{4}{x+\lambda} - \frac{4x^3}{x^4+\lambda^4} \\ &= \frac{4(x^4+\lambda^4) - 4x^3(x+\lambda)}{(x+\lambda)(x^4+\lambda^4)} \\ &= \frac{4\lambda^4 - 4x^3\lambda}{(x+\lambda)(x^4+\lambda^4)} \\ &= \frac{4\lambda(\lambda^3 - x^3)}{(x+\lambda)(x^4+\lambda^4)} \\ &= \frac{4\lambda(\lambda - x)(\lambda^2+\lambda x + x^2)}{(x+\lambda)(x^4+\lambda^4)}\end{aligned}$$

Multiply both sides by y.

$$\begin{aligned}\frac{dy}{dx} &= \frac{4\lambda(\lambda - x)(\lambda^2 + \lambda x + x^2)}{(x+\lambda)(x^4+\lambda^4)}y \\ &= \frac{4\lambda(\lambda - x)(\lambda^2 + \lambda x + x^2)}{(x+\lambda)(x^4+\lambda^4)}\left[\frac{(x+\lambda)^4}{x^4+\lambda^4}\right] \\ &= \frac{4\lambda(\lambda - x)(\lambda^2 + \lambda x + x^2)(x+\lambda)^3}{(x^4+\lambda^4)^2}\end{aligned}$$